$f : U \to \mathbb{R}^m$ with $U \subset \mathbb{R}^n$. **Differentiable wrt i th element** if following limit exists:

$$D_i f(a) = \lim_{h\to 0} \frac{1}{h}(f(a_1, \ldots, a_i + h, \ldots, a_n) - f(a_1, \ldots, a_n))$$

Jacobian of $\mathbf{f} = (f_1, \ldots, f_m)$: $J\mathbf{f}(a) = \begin{bmatrix} D_1\mathbf{f}(a) & \ldots & D_n\mathbf{f}(a) \end{bmatrix} = \begin{bmatrix} D_1 f_1(a) & \ldots & D_n f_1(a) \\ \vdots & & \vdots \\ D_1 f_m(a) & \ldots & D_n f_m(a) \end{bmatrix}$ where

$D_1\mathbf{f}(a)$ part.derivative wrt first element, $D_1 f_1(a)$ (and so on) found by eq. above.

f diff at $a \Rightarrow$ all partial derivatives of f at a exists. Note: in practice $\Leftrightarrow$

f differentiable at $a \in U$ and L derivative $\mathbf{f}$ at a ($D\mathbf{f}(a)$) if $\exists! L : \mathbb{R}^n \to \mathbb{R}$ linear s.t. $\lim_{h\to 0} \frac{1}{|h|}(\mathbf{f}(a+h) - \mathbf{f}(a) - L(h)) = 0$.

Directional derivative of v at a $\lim_{h\to 0} \frac{\mathbf{f}(a+hv)-\mathbf{f}(a)}{h}$

MVT $[a, b] \subset U$ then $\exists c \in [a, b]$ s.t. $f(b) - f(a) = (Df(c))(b - a)$

$\mathbf{f}$ diff at $a \in U \Rightarrow \begin{cases} \text{all partial derivatives } \mathbf{f} \text{ at } a \text{ exists} \\ \mathbf{f} \text{ cont. at } a \\ \text{all directional derivatives at } a \text{ exists in direction } v : D\mathbf{f}(a)v \end{cases}$

Usefull examples:

$S(A) = A^2 \Rightarrow DS(A)(H) = AH + HA$

$f(A) = A^{-1} \Rightarrow Df(A)(H) = -A^{-1}HA^{-1}$.

$f_1, \ldots, f_m : U \to \mathbb{R}$ diff at a, then $\mathbf{f}$ diff at a with $(D\mathbf{f}(a))v = \begin{bmatrix} Df_1(a)v \\ \vdots \\ Df_m(a)v \end{bmatrix}$

$\{v_1, \ldots, v_n\}$ span V if $\forall v \in V, v = \sum_i a_i v_i$. $\{v_1, \ldots, v_n\}$ basis V if span lin.indep.

Lin. Transformation $T : V \to W : T(av_1 + bv_2) = aT(v_1) + bT(v_2) \quad \forall a, b \in \mathbb{R}, \forall v_1, v_2 \in V$

Let $\{v_1, \ldots, v_n\}$ and $\{v'_1, \ldots, v'_n\}$ bases of V. sim. $\{w_1, \ldots, w_m\}$ and $\{w'_1, \ldots, w'_m\}$ of W.

$T_{v,w}$ matrix of $T : V \to W$ wrt bases v and w. With $T_{v_k} = \sum_{i=1}^{m} t_{l,k} w_l$.

Def. $v'_i = p_{1,i}v_1 + \ldots + p_{n,i}v_n$. Then Change of basis matrix is $P_{v'\to v} = \begin{bmatrix} p_{1,1} & \ldots & p_{1,n} \\ \vdots & \ddots & \vdots \\ p_{n,1} & \ldots & p_{n,n} \end{bmatrix}$

$T_{v',w'} = P^{-1}_{w'\to w} T_{v,w} P_{v'\to v}$. Let $T : V \to V$, with eig. (v, λ). $\#\lambda = \dim\{v | Tv = \lambda v\}$.

Basis for complex V is called **eigenbasis of** V for $T : V \to V$ if each element basis is eigenvector T.

Let A be $n \times n$ and $P = [v_1, \ldots, v_n]$ invertible, then 1) $\sigma(A) = \sigma(P^{-1}A)$

2) $v_1, \ldots, v_n$ eigenbasis $\mathbb{C}^n$ of $A \Leftrightarrow P$ diagonalizes A.

$A : V \to V$, with $v_1, \ldots, v_k$ eigenv. of A with distinct real eigenvalues, then $v_1, \ldots, v_k$ lin. independent.

$A \in \mathbb{C}^{n \times n}$. Then exists eigenbasis $\mathbb{C}^n$ for $A \Leftrightarrow$ eigenvalues A simple (so mult. is 1)
$U \subset \mathbb{R}^n, f : U \to \mathbb{R}^m$ diff. on U. Want to find x s.t. $f(x) = 0$.
Newton method: has initial guess x_0, solve iteratively:

$$x_{i+1} = x_i = (Df(x_i))^{-1} f(x_i)$$

$(Df(x))$ satisfies **Lipschitz condition** on $V \subset U$ with **Lipschitz ratio** M if

$$\forall x, y \in V : |(Df(x)) - (Df(y))|^{1} \leq M|x - y|$$

If $D_i f$ is itself diff. then **second partial derivative** is derivative of $D_i f$ wrt j so $D_j(D_i f)$.

Kantorovich's theorem: $x_0 \in \mathbb{R}^N$, U open. neigh. x_0 and $f : U \to \mathbb{R}^n$ diff., $Df(x_0)$ invertible.

$$h_0 = -(Df(x_0))^{-1} f(x_0) \quad x_1 = x_0 + h_0 \quad U_1 = B_{|h_0|}(x_0)$$

If $\overline{U_1} \subset U$ and der. $Df(x)$ satisfies Lipschitz condition $\forall u_1, u_2 \in \overline{U_1}$ and

$$|f(x_0)||Df(x_0)^{-1}|^2 M \leq \frac{1}{2}$$

Then $f(x) = 0$ solution in $\overline{U_1}$ and newton method converges with in. guess x_0.

Function **Strictly monotone** if $\forall x < y$ one has either $f(x) < f(y)$ (increasing) OR $f(x) > f(y)$ (decreasing).
Let $f : \mathbb{R}^n \to \mathbb{R}^n$.

Inverse function theorem 1D: $n = 1$, so $f : [a, b] \to [c, d]$ continuous with $f(a) = c, f(b) = d$ with f strictly monotone on $[a, b]$ then
-) $\exists g : [c, d] \to [a, b]$ continuous s.t. $\begin{cases} f(g(y)) = y & \forall y \in [c, d] \\ g(f(x)) = x & \forall x \in [a, b] \end{cases} \Rightarrow g$ inverse function f
-) can find $g(y)$ by solving $y - f(x)$ for x by bisection method.
-) f diff at $x \in (a, b)$ and $f'(x) \neq 0 \Rightarrow g$ diff at $f(x)$ and $g'(f(x)) = \frac{1}{f'(x)}$

Inverse function theorem f cont. diff, $Df(x_0)$ invert. Then f locally invertible with diff. inverse in some neighb. U_{x_0}. Working definition: if Df invertible, mapping locally invertible. In practice, check by determinant Jacobian.

Implicit function: function of the form $f(x_1, \ldots, x_n) = 0$

The Implicit function theorem $U \subset \mathbb{R}^n$ open, $c \in U$, $f : U \to \mathbb{R}^{n-k}$ be C^1 s.t. $f(c) = 0$, $Df(c)$ is onto [2] $\Rightarrow$ system of lin. equations $(Df(c))(x) = 0$ has $n - k$ passive and k active variables. Exists neighb. of c in which $f = 0$ implictly defines $n - k$ passive variables as a functoin g of the $k-$ active variables.

Graph: $f : \mathbb{R}^k \to \mathbb{R}^{n-k}$ given by $\Gamma(f) = \{(x, y) \in \mathbb{R}^n | f(x) = y\}$.
Manifold: $M \subset \mathbb{R}^n$ is **Smooth k-dim.** if locally it is the graph of a C^1 mapping $f : \mathbb{R}^k \to \mathbb{R}^{n-k}$ Working definition: object where derivative is well-defined.

Locus of function $f : X \to Y$ is $\{x \in X | f(x) = 0\}$

$U \in \mathbb{R}^n$ open, $f : U \to \mathbb{R}^{n-k}$ be C^1. Def. $M \subset \mathbb{R}^n : M \cap U = \{z \in U | f(z) = 0\}$.$Df(z)$ is onto $\forall z \in M \cap U \Rightarrow M \cap U$ is **Smooth k-dim. manifold embedded in** $\mathbb{R}^n$. If holds $\forall z \in M$ in such U, then M is k-dim. manifold.

M smooth $k-$ dim. manifold embedded in $\mathbb{R}^n$ $:\Rightarrow$ $\forall z \in M$, has neighb. $U \in \mathbb{R}^n$: $\exists C^1 : f : U \to \mathbb{R}^{n-k}$ with $Df(z)$ onto and $M \cap U = \{z \in M \cap U | f(z) = 0\}$
$M \subset \mathbb{R}^m$ k-dim. manifold, $U \subset \mathbb{R}^n$ open, $f : U \to \mathbb{R}^m$ be C^1 which derivative is surjective at every point $x \in f^{-1}(M)$. Then $f^{-1}(M)$ submanifold of $\mathbb{R}^n$ of dim. $k + n - m$.

$g : \mathbb{R}^n \to \mathbb{R}^n : g(x) = Ax + c$ with A inv. If $M \subset \mathbb{R}^n$ is smooth k-dim. manifold $\Rightarrow$ $g(M)$ is smooth k-dim. manifold.

Parameterization $k-$ dim. manifold $M \subset \mathbb{R}^n$ mapping $\gamma : U \subset \mathbb{R}^k \to M$ satisfies:
1) U open
2) γ is C^1, one-to-one [3] and onto M.
3) $D_\gamma(u)$ is one-to-one $\forall u \in U$.
$M \subset \mathbb{R}^n$ be $k-$ dim. Tangent space to M at $z_0 = (x_0, y_0)$ denoted $T_{z_0}M$ is graph of lin. transf. $Df(x_0)$.

$$\begin{cases} y = g(x) \text{ any point in } C\ (a, g(a)) \\ \text{tangent line } y = g(a) + g'(a)(x-a) \\ \textbf{Tangent space } dy = g'(a)dx \end{cases} \quad \& \quad \begin{cases} z = f(x,y) \text{ and } (a,b,c) \in S \\ \text{Tangent plane } z = c + Df(a,b) \begin{bmatrix} x-a \\ y-b \end{bmatrix} \\ \text{Tangent space } dz = Df(a,b) \begin{bmatrix} dx \\ dy \end{bmatrix} \end{cases}$$

$U \subset \mathbb{R}^n$ open,$F : U \to \mathbb{R}^{n-k} \in C^1$. If $F(z) = 0$ describes manifold M, $DF(z_0)$ is onto for some $z_0 \in M$, then $T_{z_0}M \in \ker(Df(z_0))$

Let $f : U \to \mathbb{R}$ with $U \subset \mathbb{R}^n$. $D_i f$ diff $\Rightarrow$ $D_i D_j f = D_j D_i f$.

Taylor polynomial multivariable: $p^k_{f,a}(a + h) = \sum\limits_{m=0}^{k} \sum\limits_{I \in \mathcal{I}^m_n} \frac{1}{I!} D_I f(a) h^I$
$u \in U$ **critial point** if $Df(u) = 0$ then $f(u)$ **critical value.**

$u \in U$ local minimum/maximum $\Rightarrow Df(u) = 0$.
f of class C^2, x_0 crit. point. If $f"(x_0) > 0$ then x_0 strict local minimum of f.

u critical point, f of C^2, then $Q_{f,u} : \mathbb{R}^n \to \mathbb{R}$ s.t. $Q_{f,u}(h) := \sum\limits_{I \in \mathcal{I}_n^2} \frac{D_I f(u)}{I!} h^I$

Quadratic form in n variables: $Q(x) = \sum\limits_{I \in \mathcal{I}_n^2} a_I x^I$. **Matrix of Q** $Q_A(x) = x^T A x$

$\{$Quadratic forms in n variables$\} \leftrightarrow \{A \in \text{Mat}(n, n) : A^T = A\}$
Q, Q' are **equivalent** if $\exists S \in \text{Mat}(n, n)$ nonsing. s.t. $Q'(x) = Q(Sx)$
Q, Q' with A, A' then Q, Q' equivalent if $\exists S : A' = S^T A S$
Q diag. if $Q(x) = \sum\limits_{i=1}^{n} a_i x_i^2 \Leftarrow A_{i,i} = a_i$.
Hubbard theorem:
-) $\forall Q, \exists S \in \text{Mat}(n, n)$ s.t. $Q(Sx)$ is diag.
-) **Signature** (k, l) with $k = \#\{a_i : a_i > 0\}$ and $\#\{a_i : a_i < 0\}$ is same for all diag. matrices.

Rank $k + 1 \leq n$. Q is **non-degenerate** if $k + l = n$. Equiv. A is nonsing. Otherwise Q **Degenerate.u** is **Saddle** if $k, l > 0$.
Let Q, Q' be non-deg. with equal sign. Then Q, Q' are equiv. as quadratic forms: $Q'(x) = Q(Sx)$ for some $S \in \text{GL}(n, \mathbb{R})$ **Hessian matrix** Matrix $Q_{f,u}$ multiplied by 2.
Hubbard props:
1) f of class C^2, u crit. point, then
-) $Q_{f,u}$ positive definite (i.e. $Q(x) > 0, \forall x \neq 0$) $\Rightarrow u$ is strict local min.
-) $Q_{f,u}$ negative definite (i.e. $Q(x) < 0, \forall x \neq 0$) $\Rightarrow u$ is strict local max.
2) $U, V \subset \mathbb{R}^n$ open,C^2 mapping $(V, y_0) \xrightarrow{\varphi} (U, x_0) \xrightarrow{f} \mathbb{R}$ supp. x_0 crit. point f then:
-) y_0 crit. point $f \circ \phi$
-) $T = D\varphi(y_0)$ invertible $\Rightarrow$ signature y_0 equals sign. x_0.
3) $f \in C^2(U, \mathbb{R})$ if $u \in U$ saddle $\Rightarrow \forall$ neighb. V of u, $\exists a, b \in V$ s.t. $f(a) < f(u) < f(b)$.

n-th order ODE: $F(t, x, \ldots, x^{(n)}) = 0$.**explicit form:** $x^{(n)} = f(t, \ldots, x^{(n-1)})$
Explicit n-th order ODEs transformed into system n first oder ODEs:

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} x \\ \vdots \\ x^{(n-1)} \end{bmatrix} \Rightarrow \dot{u} = \begin{bmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_n \end{bmatrix} = \begin{bmatrix} u_2 \\ \vdots \\ f(t, u_1, \ldots, u_n) \end{bmatrix}$$

$x(t) = (x_1(t), \ldots, x_n(t))^T$.**initial condition** value at in. time (t_0) given by $x_0 = x(t_0)$
Particular solution: solution satisfying init. condi. **general:** solution n-th order ODE, contains n param.**Complete:** all particular solutions can be obtain. by general.
Singular: part. solution not obtained from general.

x^* **equilibrium points** if $f(x^*) = 0$, solution ODE. **Phase space** Space $X \subset \mathbb{R}^n$ where sol. defined. $\dot{x} = f(x), \forall x \in X$. The set of all vel. vector is **vector field.**. Curve tangent to vector field: **integral curve.**. Set of all integral curves: **flow**

$(t, x) \in U \subset \mathbb{R} \times \mathbb{R}^n$. If U,open $f : U \to \mathbb{R}^n$ continuous, then U **extended phase-space** f time dep. vector field, $\dot{x} = f(t, x)$ is **nonautonomous**
$U = \mathbb{R} \times \tilde{U}, \tilde{U} \subseteq \mathbb{R}^n$ open, $f(t, x) = \tilde{f}(x) \Rightarrow$ ODE **autonomous**.
Diff. $\phi : I \to \mathbb{R}^n, I \subseteq \mathbb{R}$ sol. if

$$\text{graph}(\phi) := \{(t, \phi(t)) | t \in I\} \subset U \qquad \left.\frac{d\phi}{dt}\right|_{t=\tau} = f(t, \phi(\tau)), \tau \in I$$

Time dep. vector field $f : U \to \mathbb{R}^n$ satisf.

Global Lipschitz $L \in \mathbb{R}$ if $\|f(t, x_0) - f(t, x_1)\| \leq L|x_0 - x_1\| \quad (t, x_i) \in U, i = 0, 1$

Local Lipschitz cond. $\forall (T, x) \in U$, exists neighb. $V \subseteq U$ s.t. $f|_V$ satisfies global Lipschitz in V for $L = L(t, x) \in \mathbb{R}$

Paeno theorem existence solutions IVP:
$f : U \to \mathbb{R}^n$ cont $\Rightarrow \forall (t_0, x_0) \in U, \exists \Delta t > 0$ small and sol. $\phi : [t_0 - \Delta t, t_0 + \Delta t] \to \mathbb{R}^N$ of ODE $\dot{x} = f(t, x)$ satisfying $\phi(t_0) = x_0$.
Uniqueness solutions IVP:
$\dot{x} = f(t, x)$. Solution $\phi(t)$ to IVP is uniq.if 2 sol. $\phi_1 : I_1 \to \mathbb{R}^n, \phi_2 : I_2 \to \mathbb{R}^n$ conincide on the interval $I = I_1 \cap I_2$
Picard Lindelof, uniq. of sol. IVP
If $f : U \to \mathbb{R}^N$ satisfy. Lip. cond in $U \Rightarrow \forall (t_0, x_0) \in U, \exists \varepsilon > 0$ s.t.
$\dot{x} = f(t, x), x(t_0) = x_0$ has unique solution $\phi : [t_0 - \varepsilon, t_0 + \varepsilon] \to \mathbb{R}^n$.
In proof, Banach's fixed point theorem: (M, d) complete MS, $f : M \to M$ s.t. $d(f(q), f(p)) \leq kd(q, p), \forall q, p \in M$ with $0 \leq k < 1$. Then f uniq. fixed point $p_0 \in M$: $f(p_0) = p_0$, $f(p) = p \Rightarrow p = p_0$.
Lemma:
$f : U \to \mathbb{R}^n$ cont. diff $\Rightarrow$ lip. cond. satisfied on every compact & convex
$V \subseteq U$ with $L := \sup_{(t,x) \in V} \|D_x f(t, x)\|$
Theorem:
f satisfies Lipschitz in $U \Rightarrow \forall (t_0, x_0) \in U$ ex. neighb. $V \subset U$ and interval
$I_\varepsilon = [-\varepsilon, \varepsilon]$ s.t. $\phi : I_\varepsilon \times V \to U : (s, (t_0, x_0)) \mapsto \phi(t_0 + s)$ sol. to IVP cont. mapping.

$\dot{x} = Ax$ then sol. $x(t) = \exp(tA)x(0)$.
$v_1, \ldots, v_{n_s}, w_1, \ldots, w_{n_c}, u_1, \ldots, u_{n_u}$ gen. eig. associated to the n_s, n_u, n_c eigenv. with negative,positive,zero real parts. $E^s = \text{span}\{v_1, \ldots, v_{n_s}\}, E^u = \text{span}\{u_1, \ldots, u_{n_u}\}$

$E^c = \text{span}\{w_1, \ldots, w_{n_c}\}$ are stable, unstable, center subspaces.

$A \in \mathbb{R}^{n\times n}$ is **Hyperbolic** if $\text{Re}(\lambda) \neq 0, \forall \lambda \in \sigma(A)$ then $\dot{x} = Ax, x^*$ also hyperbolic. $\text{ind}(A)$ sum algeb. multip. of eigenvalues A with neg. real part.

$\dot{x} = Ax$ hyperbolic $\Rightarrow \exists!$ decomp. $\mathbb{R}^n = E^s \bigoplus E^u$, both invariant under flow. $\phi|_{E_s} \xrightarrow{t\to\infty}$ origin, and $\phi|_{E_u} \xrightarrow{t\to-\infty}$ origin. $\dim(E^s) = \text{ind}(A)$
A, B hyperbolic. $\exists$ time direction preserving homeomorphism, $h : \mathbb{R}^n \to \mathbb{R}^n$ mapping solutions system A to sol. system $B \Leftrightarrow A, B$ same index. I.e. systems **topologically equiv.**
Seperation of variables:
$\dot{x} = g(x)f(t)$ with $g : U \to \mathbb{R} \in C^0, f : V \to \mathbb{R} \in C^0$. Then IVP has unique local sol. obtained by solving $\int\limits_{x_0}^{x} \frac{dx}{g(x)} = \int\limits_{t_0}^{t} f(t)dt$ for x as func. t.

$\dot{x} = A(t)x$, **Fundamental matrix M(t)** $n \times n$ matrix, with columns lin. indep. sol IVP. **Principal fundamental matrix** $M(t_0) = I$. If $A(t)$ constant, then $M = \begin{bmatrix} e^{\lambda_1 t}v_1 & \ldots & e^{\lambda_n t}v_n \end{bmatrix}$. $M(t)$ fundamental $\Rightarrow$ nonsingular $\Rightarrow \dot{M}(t) = A(t)M(t)$. So $x(t) = M(t)M^{-1}(t_0)x_0$ solves IVP.

Variation of constants:
$\dot{x} = A(t)x + b(t), M(t)$ principal fund. matrix $\dot{x} = A(t)x$. Ass. $M(t)c(t)$ general solution inhomogoneous system. After some algebra we get

$$x(t) = M(t)\left(c(t_0) + \int\limits_{t_0}^{t} M^{-1}(s)b(s)ds\right)$$

$\dot{x} = f(x), x \in \mathbb{R}^n$, so $\begin{array}{c} \dot{x_1} = f_1(x_1, \ldots, x_n) \\ \vdots \\ \dot{x_n} = f_n(x_1, \ldots, x_n) \end{array}$. Map is C^k**-vector field.** Another way:
$v : U \to U \times \mathbb{R}^n, x \mapsto (x, f(x)) =: v(x)$
$f = f(x) \in C^1$. Def. $\phi_t(x) := (-\varepsilon, \varepsilon) \times U \to \mathbb{R}^n$ **flow of** f **through** x_0 as unique solution $\dot{x} = f(x), \phi_0(x) = x_0$.
Proposition:
ODE $\dot{x} = f(x)$ autonomous. $|t|, |s|, |t + s| < \varepsilon, \Rightarrow \phi_{t+s}(x) = \phi_t(\phi_s(x)) = \phi_s(\phi_t(x))$

Support of $f = \overline{\{x \in \mathbb{R}^n | f(x) \neq 0\}}$. **Oscillation** $\text{osc}_A(f) = M_A(f) - m_A(f) = \sup\limits_{x\in A} f(x) - \inf\limits_{x\in A} f(x)$. f bounded support: $\exists R > 0 : f(x) = 0$ for $|x| > R$
Dyadic cube $C_{k,N} = \left\{x \in \mathbb{R}^n | \frac{k_i}{2^N} \leq x_i < \frac{k_{i+1}}{2^N}, 1 \leq i \leq n\right\}$. $N-$

th dyadic paving ,of $\mathbb{R}^n$ den. $\mathcal{D}_N(\mathbb{R}^n) = \{C_{k,N}|N\}$.$\text{vol}_n C = \frac{1}{2^{Nn}}$
Lower and upper integral limit of **Lower and upper sums**

$$U(f) = \lim_{N\to\infty} U_N(f) = \lim_{N\to\infty} \sum_{C\in\mathcal{D}_N} M_C(f)\text{vol}_n C = \lim_{N\to\infty} \frac{1}{2^{Nn}} \sum_{C\in\mathcal{D}_n} M_C(f)$$

$$L(f) = \lim_{N\to\infty} L_N(f) = \lim_{N\to\infty} \sum_{C\in\mathcal{D}_N} m_C(f)\text{vol}_n C = \lim_{N\to\infty} \frac{1}{2^{Nn}} \sum_{C\in\mathcal{D}_n} m_C(f)$$

f **Integrable** if $U(f) = L(f) \Rightarrow \int\limits_{\mathbb{R}^n} f|d^n x| = U(f) = L(f)$.
f integ $\Rightarrow$ Choose $x_{k,N} \in C_{k,n}$ consider $R(f,N) = \sum\limits_{k\in\mathbb{Z}^n} \text{vol}_n(C_{k,N})f(x_{k,N})$. Since f ,bounded, so $m_{C_{k,N}}(f) \leq f(x_{k,N}) \leq M_{C_{k,N}}(f)$ so convergence.

$f_1(x)$ integ. on $\mathbb{R}^n$, $f_2(y)$ integ. on $\mathbb{R}^m \Rightarrow g(x,y) = f_1(x)f_2(y)$ integ. on $\mathbb{R}^{n+m}$ with $\int\limits_{\mathbb{R}^{n+m}} g|d^n x||x^m y| = \left(\int\limits_{\mathbb{R}^n} f_1|d^n x|\right)\left(\int\limits_{\mathbb{R}^m} f_2|d^m y|\right)$
$A \subset \mathbb{R}^n$ bounded, $\mathbf{1}_A$ integrable, then $\text{vol}_n A := \int\limits_{\mathbb{R}^n} \mathbf{1}_A|d^n x|$. Set is **pavable** if well-defined volume ($\mathbf{1}_A$ is integ.)
$n-$ dim. paralel. $P = I_1 \times \cdots \times I_n$ s.t. $I_i = [a_i, b_i] \Rightarrow \text{vol}_n(P) = |b_1 - a_1| \ldots |b_n - a_n|$
Let $A, B \subset \mathbb{R}^n, v \subset \mathbb{R}^n, t \in \mathbb{R}$ with A, B disjoint both pavable.
1) $\text{vol}_n(A \cup B) = \text{vol}_n(A) + \text{vol}_n(B)$. 2) $A + v$ pavable,$\text{vol}_n(A + v) = \text{vol}_n(A)$.3) A has volume $\Rightarrow \text{vol}_n(tA) = |t|^n \text{vol}_n(A)$.
$X \subset \mathbb{R}^n$ bounded volume zero iff $\forall\varepsilon > 0, \exists N$ s.t. $\sum\limits_{\substack{C\in\mathcal{D}_N(\mathbb{R}^n)\\ C\cap X\neq\emptyset}} \text{vol}_n(C) \leq \varepsilon$
f integrable $\Leftrightarrow$ bnd with bnd support, $\forall\varepsilon > 0, \exists N : \sum\limits_{\{C\in\mathcal{D}_N(\mathbb{R}^n)|\text{osc}_C(f)>\varepsilon\}} \text{vol}_n(C) < \varepsilon$
f integrable,$\Gamma(f)$ graph, $C_0 \subset \mathbb{R}^n \Rightarrow \text{vol}_{n+1}(\Gamma(f) \cap (C_0 \times \mathbb{R})) = 0$
$M \subset \mathbb{R}^n$ manifold of dim. $k < n$.$\forall X \subset M$ bounded, $\text{vol}_n(X) = 0$.
f,continuous and bnd support $\Rightarrow$ integrable.
$X \subset \mathbb{R}^n$,compact, $f : X \to \mathbb{R}$ cont. $\Gamma(f) \subset \mathbb{R}^{n+1}$ has vol. 0. $U \subset \mathbb{R}^n, f : U \to \mathbb{R}$ cont. then $\text{vol}_{n+1}(Y) = 0$ where $Y \subset \Gamma(f)$ compact.
f bnd, bnd. support integrable $\Leftrightarrow f$ cont. except on set of volume zero.
$X \subset \mathbb{R}^n$,**measure zero** $\forall\varepsilon > 0, \exists$ inf. open boxes $B_i : X \subset \cup B_i, \sum \text{vol}_n(B_i) \leq \varepsilon$.
$i \mapsto X_i$ seq. set of measure $0 \Rightarrow \cup X_i$ measure zero. f bnd, bnd supp. f integralbe $\Leftrightarrow$ cont. except on set measure zero.
Fubini's theorem $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ int. function. Suppose $\forall x \in \mathbb{R}^n, y \mapsto f(x,y)$ int $\Rightarrow$ $x \mapsto \int\limits_{\mathbb{R}^m} f(x,y)|d^m y|$ integ. and $\int\limits_{\mathbb{R}^{n+m}} f(x,y)|d^n x||d^m y| = \int\limits_{\mathbb{R}^n} \left(\int\limits_{\mathbb{R}^m} f(x,y)|d^m y|\right)|d^n x|$

$T : \mathbb{R}^n \to \mathbb{R}^n$ lin. trans. $[T]$ ass. matrix. Then for any $A \subset \mathbb{R}^n$ pavable $\Rightarrow$ $T(A)$ pavable $\Rightarrow \text{vol}_n T(A) = |\det[T]| \text{vol}_n A = |\det[T]| \int\limits_{\mathbb{R}^N} \mathbf{1}_A |d^n x|$.

k-paralellogram $P(v_1, \ldots, v_k) = \{t_1 v_1 + \ldots + t_n v_n | 0 \leq t_i \leq 1, v_i \in \mathbb{R}^n\}$.
unit n-dim. cube $Q_n = P(e_1, \ldots, e_n)$.**Volume:** $\text{vol}_n P(v_1, \ldots, v_n) = |\det[v_1, \ldots, v_n]|$.
Theorem lin. change: $T : \mathbb{R}^n \to \mathbb{R}^n$ inv. lin. trans, $f : \mathbb{R}^n \to \mathbb{R}$ int.
$\Rightarrow f \circ T$ int.& $\int\limits_{\mathbb{R}^n} f(y)|d^n y| = |\det T| \int\limits_{\mathbb{R}^N} f(T(x))|d^n x|$
Theorem nonlin. change: $X \subset \mathbb{R}^n$ compact, ∂X volume zero. $X \subset U$ with U open. $\phi : U \to \mathbb{R}^n \in C^1$ injective on $X \setminus \partial X$ and Lipschitz derivative, with $[D\phi(x)]$ inv.$\forall x \in X \setminus \partial X$. Set $Y = \phi(X)$. Then $f : Y \to \mathbb{R}$ integ., $(f \circ \phi)|\det[D\phi]|$ int. on X and $\int\limits_Y f(y)|d^n y| = \int\limits_X (f \circ \phi)|\det[D\phi(x)]||d^n x|$
Prop: $v_1, \ldots, v_k \in \mathbb{R}^k$,s.t. $T = [v_1, \ldots, v_k] \in \mathbb{R}^{k \times k} \Rightarrow \text{vol}_k P(v_1, \ldots, v_k) = \sqrt{\det(T^T T)}$
$T = [v_1, \ldots, v_k] \in \mathbb{R}^{n \times k}$ **k-dim volume** $P(v_1, \ldots, v_k)$ is $\text{vol}_k P(v_1, \ldots, v_k) := \sqrt{\det(T^T T)}$
parallelogram in $\mathbb{R}^n$ **anchored at** $x \in \mathbb{R}^n$ is $P_x(v_1, \ldots, v_k) \Rightarrow \text{vol}_k P(v_1, \ldots, v_k) = \text{vol}_k P_x(v_1, \ldots, v_k)$.
$\text{vol}_k(X) = 0$ for arbitrary $X \subset \mathbb{R}^n$ if $\forall R$ the bounded $X \subset B_R(0)$ has k-dim. volume 0.
$0 \leq m < k \leq n$. If $M \subset \mathbb{R}^N$ is m-dim. manifold, $X \subset M$ closed,$\text{vol}_k(X) = 0$.

$M \subset \mathbb{R}^n$ be k-dim. manifold,$U \subset \mathbb{R}^k$ with boundary $k-$ dim. volume 0.
Let $X \subset U$ s.t. $U \setminus X$ open. $\gamma : U \to \mathbb{R}^n$ cont. **parametrizes** M if:
1) $M \subset \gamma(U)$
2) $\gamma(U \setminus X) \subset M$
3) $\gamma : U \setminus X \to M$ is one-to-one and C^1.
4) $D\gamma(u)$ is one-to-one,$\forall u \in U \setminus X$.
5) $\text{vol}_k(X) = 0, \text{vol}_k(\gamma(X) \cap C) = 0,$ for $C \subset M$ compact.
ϕ def. domain in $\mathbb{R}^n$ by U, we denote pair as $[\phi(U)]$.
Theorem: All manifolds can be parametrized.
Theorem $M \subset \mathbb{R}^n$ smooth k-dim. $U \subset \mathbb{R}^k$, pavable, $\gamma : U \to M$ parametrization, X all bad points then:

1. $$\text{vol}_k := \int\limits_{\gamma(U \setminus X)} |d^k x| = \int\limits_{U \setminus X} \left(|d^k x|(P_{\gamma(u)}(D_1\gamma(u), \ldots, D_k\gamma(u)))\right) |d^k u|$$
$$= \int\limits_{U \setminus X} \sqrt{\det([D\gamma(u)]^T [D\gamma(u)])} |d^k u|$$

2. $f : M \to \mathbb{R}$ **Integrable over** M **wrt volume if**

$$\int_M f(x)|d^k x| := \int_{U\setminus X} f(\gamma(u))\sqrt{\det([D\gamma(u)]^T[D\gamma(u)])}|d^k u|$$

Exists. In particular $Y \subset M$ s.t. $\mathbf{1}_{\gamma^{-1}(Y)}$ integrable then

$$\text{vol}_k Y = \int_{U\setminus X} \mathbf{1}_Y(\gamma(u))\sqrt{\det([D\gamma(u)]^T[D\gamma(u)])}|d^k u|$$

M k-dim. manifold $\mathbb{R}^n$, $f : M \to \mathbb{R}$ function, $U, V \subset \mathbb{R}^k$. $\gamma_1 : U \to M$
$\gamma_2 : V \to M$ parametrizations M then

$$\int_U f(\gamma_1(u))\sqrt{\det([D\gamma_1(u)]^T[D\gamma_1(u)])}|d^k u| \quad \text{exists iff}$$

$$\int_V f(\gamma_2(v))\sqrt{\det([D\gamma_2(v)]^T[D\gamma_2(v)])}|d^k v| \quad \text{exists} \Rightarrow \text{integrals equal}$$

Constant k-form on $\mathbb{R}^n$ function $\phi : \underbrace{\mathbb{R}^n \times \ldots \mathbb{R}^n}_{k-\text{times}} \to \mathbb{R}$ takes $k-$ vectors, returns number $\phi(v_1, \ldots, v_k)$ s.t. multi-linear ($\phi(v_1, \ldots, au+bw, \ldots, v_k) = a\phi(v_1, \ldots, u, \ldots, v_k)+ b\phi(v_1, \ldots, w, v_k)$) and antisymmetric (chaging 2 elements changes the sign). k is degree of form.

$a, b \in \mathbb{R}^3 \Rightarrow dx_1 \wedge dx_2(a, b) = \det \begin{bmatrix} a_1 & b_1 \\ 2_2 & b_2 \end{bmatrix}$.

We notice $dx_1 \wedge dx_2$ =**signed area** $P(\Pi_{x_1,x_2}(a), \Pi_{x_1,x_2}(b))$

$dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ on $(v_1, \ldots, v_k)$ gives $(x_{i_1}, \ldots, x_{i_k})$ comp. of signed $K-$ dim. volume of $k-$ parallelogram spanned by $v_1, \ldots, v_k$.

Elementary k-form $dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ s.t. $1 \le i_1 < \cdots < i_k \le n$.

Note: $(\phi+\psi)(v_1, \ldots, v_k) = \phi(v_1, \ldots, v_k)+\psi(v_1, \ldots, v_k)$ and $(a\phi)(v_1, \ldots, v_k) = a(\phi(v_1, \ldots, v_k))$

$A^k_c(\mathbb{R}^n)$ space constant k-forms on $\mathbb{R}^n$. $\dim(A^k_c(\mathbb{R}^n)) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$

El. cons. k-form, forms basis $A^k_c(\mathbb{R}^n)$ every k-form uniquely written
$\phi = \sum_{1\le i_1<\ldots<i_k\le n} a_{i_1\ldots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ with $a_{i_1\ldots i_k} = \phi(e_{i_1}, \ldots, e_{i_k})$

Wedge product $\phi \in A^K_c(\mathbb{R}^n)$ and $\psi \in A^l_c(\mathbb{R}^n)$ is

$$(\phi \wedge \psi)(v_1, \ldots, v_{k+l}) := \sum_{\sigma\in \text{perm}(k,l)} \text{sgn}(\sigma)\phi(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\psi(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)})$$

with $\sigma(1) < \ldots < \sigma(k)$ and $\sigma(k+1) < \ldots < \sigma(k+l)$

$\phi \wedge (\psi_1 + \psi_2) = \phi \wedge \psi_1 + \phi \wedge \psi_2$. $(\phi_1 \wedge \phi_2) \wedge \phi_3 = \phi_1 \wedge (\phi_2 \wedge \phi_3)$. ϕ k-form,$\psi-$ l-form, then $\phi \wedge \psi = (-1)^{kl} \psi \wedge \phi$.
k-form field on $U \subset \mathbb{R}^n$ is $\phi : U \to A_c^k(\mathbb{R}^n)$. Space: $A^k(U)$.

$U \subset \mathbb{R}^n$ bounded open,$\text{vol}_k \partial U = 0$. $V \subset \mathbb{R}^n$ open,$[\phi(U)]$ param. domain in V. ϕ a k-field. Then **integral of** ϕ **over** $[\phi(U)]$ **is**

$$\int_{[\phi(U)]} \phi := \int_U \phi(P_{\phi(u)}(D_1\phi(u), \dots, D_k\phi(u)))|d^k u|$$

V fin. dim. real vector space, $\mathcal{B}_V$ set of bases,V. **Orientation** V is $\Omega : \mathcal{B}_V \to \{+1, -1\}$. S.t. if $\{v\}, \{w\} \in \mathcal{B}_V$ with matrix $[P_{w \to v}]$ then $\Omega(\{w\}) = \text{sgn}(\det[P_{w\to v}])\Omega(\{v\})$.
$\{u\} \in \mathcal{B}_V$ **direct** if $\Omega(\{u\}) = 1$.**indirect** if $\Omega(\{u\}) = -1$.
Orientation specified by $\{v\}$ is $\Omega^{\{v\}}$, i.e. orientation for which $\{v\}$ is direct.**Standard orientation** on $\mathbb{R}^n$ is Ω^{st} by choosing standard basis as direct.
$M \subset \mathbb{R}^n$ k-dim. man., $\mathcal{B}(M)) = \left\{(x, v_1, \dots, v_k) \in \mathbb{R}^{n(k+1)}\right\}$ with $x \in M, v_1, \dots, v_k$ basis T_xM.
$\mathcal{B}_X(M) \subset \mathcal{B}(M)$ s.t. first coordinate x, so $\mathcal{B}_x(M) = \{x\} \times \mathcal{B}_{T_xM}$
orientation k-dim. manifold $M \subset \mathbb{R}^n$, is cont. $\mathcal{B}(M) \to \{+1, -1\}$ whose restriction to each $\mathcal{B}_x(M)$ is orientation T_xM.
$U \subset \mathbb{R}^n$ open,$f : U \to \mathbb{R}^{n-k} \in C^1$ s.t $[Df(x)]$ surjec. $\forall x \in M := f^{-1}(0)$. Then
$\Omega(v_1, \dots, v_k) := \text{sgn} \det[\nabla f_1(x), \dots, \nabla f_{n-k}(x), v_1, \dots, v_k]$ orientation of M.
Lin. trans $T : \mathbb{R}^k \to V$ is **orientation (p)reserving** if $\Omega(T(e_1), \dots, E_k)) = -1(+1)$
$\gamma : U \to \mathbb{R}^m$ parametrize M. Then γ is orientation preserving if
$\forall u \in (U \setminus X) : \Omega(D_1\gamma(u), \dots, D_k\gamma(u)) = +1$
$M \subset \mathbb{R}^n$ k-dim. manifold, $U_1, U_2 \subset \mathbb{R}^k, \gamma_i : U_i \to \mathbb{R}^n$ two param. M. Then
-) γ_1, γ_2 are either both orientation preserving or both orientation reserving iff
$\forall u_i \in U_i^{\text{OK}}$ with $\gamma_1(u_1) = \gamma_2(u_2)$ we have $\det[D(\gamma_2^{-1} \circ \gamma_1)(u_1)] > 0$.
-) for any $k-$ form ϕ def. on neighb. of M $\int\limits_{[\gamma_1(U_1)]} \phi = \int\limits_{[\gamma_2(U_2)]} \phi$

$M \subset \mathbb{R}^n$ k-dim. manifold,ϕ k-form field on neighb M,$\gamma : U \to M$ orientation preserving parametrization of M. Then

$$\int_M \phi = \int_{[\gamma(U)]} \phi = \int_U \phi(P_{\gamma(u)}(D_1\gamma(u), \dots, D_k\gamma(u)))|d^k u|$$

Let $\vec{F} = \begin{bmatrix} F_1 & \dots & F_n \end{bmatrix}^T$ then:

Work-form $W_{\vec{F}}(P_x(v)) = \vec{F}(x) \cdot v$. Coordinates: $F_1dx + F_2dy + F_3dz$.

Flux form on vector field in $\mathbb{R}^n$ is $\phi_{\vec{F}}(P_x(v_1, \dots, v_{n-1})) = \det[\vec{F}(x), v_1, \dots, v_{n-1}]$. Coordinates: $G_1dy \wedge dz - G_2dx \wedge dz + G_3dx \wedge dy$

Mass form $M_f(P_x(v_1,\dots,v_k)) = f(x)\det[v_1,\dots,v_k]$. Coordinates: $M_f = f dx \wedge dy \wedge dz$.

We can also integrate them:

Work of vector field $\vec{F}$ along oriented curve, C and $\gamma : [a,b] \to C$ orientation preserving param.

$$\int_C W_{\vec{F}} = \int_{[a,b]} W_{\vec{F}} P_{\gamma(u)}(D_1\gamma(u))|du| = \int_a^b \vec{F}(\gamma(u)) \cdot \gamma'(u) du$$

Flux vector field $\vec{F}$ over oriented surface S and $\gamma; U \to S$ s.t. $U \subset \mathbb{R}^2$ orientation-preserving param.

$$\int_S \phi_{\vec{F}} = \int_U \phi_{\vec{F}}(P_{\gamma(u)}(D_1\gamma(u), D_2\gamma(u)))|d^2u| = \int_U \det[\vec{F}(\gamma(u)), D_1\gamma(u), D_2\gamma(u)]|d^2u|$$

Integral M_f (or integral f $U, V \subset \mathbb{R}^3$ open, $\gamma : U \to V \in C^1$ standard orientation preserving. If $f : V \to R$ function then

$$\int_V M_f = \int_U M_f(P_{\gamma(u)}(D_1\gamma(u), D_2\gamma(u)), D_2\gamma(u))|d^3u|$$
$$= \int_U \det[\vec{F}(\gamma(u)), D_1\gamma(u), D_2\gamma(u), D_3\gamma(u)]|d^3u|$$

M k-dim. man., $X \subset M$, $\partial_M X = \{x \in M | \forall \text{neighb}\, U_x \text{ of } x, \text{points of } X \text{ and points of } M \setminus X\}$
$x \in \partial_M X$ **Smooth point of boundary if** $\exists$ neighb. V of x, unique C^1 func. $g : V \cap M \to \mathbb{R}$ s.t.1) $g(x) = 0$ 2) $X \cap V = \{g \geq 0\}$,3) $[Dg(x)] : T_x M \to \mathbb{R}$ onto.
Smooth boundary of X $\partial^s_M X$ set of all smooth points of boundary of X. (dim: $k-1$)
$M \subset \mathbb{R}^n$ k-dim.man. $X \subset M$. **Piece with boundary** of $k-$ dim man. M is $X \subset M$ compact s.t. $\text{vol}_{k-1}(\partial^s_M X) < \infty$, $\text{vol}_{k-1}(\partial_M X \setminus \partial^s_M X) = 0$.
$x \in X$ **corner point of codim.** m if $\exists V_x \subset \mathbb{R}^n$, $g : V \cap M \to \mathbb{R}^m \in C1$ s.t. 1) $g(x) = 0$ 2) $X \cap V = \{g \geq \mathbf{0}\}$,3) $[Dg(x)]$ onto.**Piece with corner:** X if $\forall x \in \partial_M X$ corner point.
Theorem: $X \subset M$ k-dim. piece with boundary $\Rightarrow \text{vol}_k(X) < \infty$

$M \subset \mathbb{R}^n$ man.$X \subset M$ piece-with-boundary. $x \in \partial_M^s X, g$ corresp.func. v tangent to M at x then v **points outward (inward)** from(to) X if $[Dg(x)]v < 0 (> 0)$
M k-dim.man., oriented by Ω, P piece with boundary M. Let $x \in \partial_M^s P, v_{\text{out}} \in T_x M$, then $\Omega^\partial : \mathcal{B}(T_x \partial P) \to \{+1, -1\}$ by $\Omega_x^\partial(v_1, \dots, v_{k-1}) := \Omega_x(v_{\text{out}}, v_1, \dots, v_{k-1})$ is orientation on $\partial_M^s P$.**prop:** $P_x(v_1, \dots, v_k)$ stand.orien.
$\partial P_x(v_1, \dots, v_k) = \sum_{i=1}^{k} (-1)^{i-1} (P_{x+v_i}(v_1, \dots, \hat{v}_i, \dots, v_k) - P_x(v_1, \dots, \hat{v}_i, \dots, v_k)$
Exterior derivative $\mathbf{d}\phi(P_x(v_1, \dots, v_{k+1})) = \lim_{h \to 0} \frac{1}{h^{k+1}} \int_{\partial P_x(hv_1, \dots, hv_{k+1})} \phi$
We work with $\mathbf{d}\phi = \sum_{1 \le i_1 < \dots < i_k \le n} \mathbf{d}(a_{i_1, \dots, i_k}) \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$
$\mathbf{d}(\mathbf{d}(\phi)) = 0, \mathbf{d}(\phi \wedge \psi) = \mathbf{d}\phi \wedge \psi + (-1)^k \phi \wedge \mathbf{d}\psi$ with ϕ k-form.
$\text{grad} f = \nabla f = \begin{bmatrix} D_1 f \\ D_2 f \\ D_3 f \end{bmatrix}, \text{curl} \vec{F} = \nabla \times \vec{F} = \begin{bmatrix} D_2 F_3 - D_3 F_2 \\ D_3 F_1 - D_1 F_3 \\ D_1 F_2 - D_2 F_1 \end{bmatrix}, \text{div} \vec{F} = \nabla \cdot \vec{F}.$
$\mathbf{d}f = W_{\nabla f} = W_{\text{grad} f}, \mathbf{d}W_{\vec{F}} = \phi_{\nabla \times \vec{F}} = \phi_{\text{curl} \vec{F}}, \mathbf{d}\phi_{\vec{F}} = M_{\nabla \cdot \vec{F}} = M_{\text{div} \vec{F}}$
$\triangle f := (D_1^2 + D_2^2 + D_3^2)(f) = \text{div}\,\text{grad} f$
$T : V \to W$ lin.trans, ϕ const. $k-$ form.**Pullback by** T is $T^* : A_c^k(W) \to A_c^k(V)$ s.t. $T^*\phi(v_1, \dots, v_k) := \phi(T(v_1), \dots, T(v_k))$
$T : \mathbb{R}^n \to \mathbb{R}^m$ with $x_1, \dots, x_n \in \mathbb{R}^n, y_1 \dots, y_m \in \mathbb{R}^m$. Then $T^*(dy_{i_1} \wedge \dots \wedge dy_{i_k}) = \sum_{1 \le j_1 < \dots < j_k \le n} b_{j_1, \dots, j_k} dx_{j_1} \wedge \dots \wedge dx_{j_k}$ wiht $b_{j_1, \dots, j_k}$ taking the matrix T with the $i_1, \dots, i_k$ rows and $j_1, \dots, j_k$ columns and taking determinant.
ϕ k-form $Y, f : X \to Y \in C^1 \Rightarrow (f^*\phi)(P_x(v_1, \dots, v_k)) := \phi(P_{f(x)}([Df(x)]v_1, \dots, [Df(x)]v_k))$
$(g \circ f)^*\phi = f^* g^* \phi, f^*\phi \wedge f^*\psi = f^*(\phi \wedge \psi)$. $\mathbf{d}f^*\phi = f^*\mathbf{d}\phi$.

[1] $|A| = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}^2}$

[2] **onto** i.e. surjective $\forall y \in \mathbb{R}^{n-k}, \exists x \in U : f(x) = y$.

[3] **one-to-one** i.e. injective $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$